Lecture 23

$\boxed{\text{Ex}}$ $W = \text{span}\left\{ \underset{\vec{v}_1}{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix}0\\1\\1\\0\end{pmatrix}}, \underset{\vec{v}_3}{\begin{pmatrix}0\\0\\1\\1\end{pmatrix}} \right\} \subset \mathbb{R}^4$

Turn into $\perp$ basis:

This is the Graham-Schmidt process

$$\vec{w}_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

$$\vec{w}_2 = \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \text{proj}_{w_1}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}-1/2\\1/2\\1\\0\end{pmatrix} \xrightarrow[\times 2]{} \begin{pmatrix}-1\\1\\2\\0\end{pmatrix}$$

$$\vec{w}_3 = \vec{v}_3 - \text{proj}_{\vec{w}_1}(\vec{v}_3) - \text{proj}_{\vec{w}_2}(\vec{v}_3)$$

$$= \begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \frac{0}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} - \frac{2}{6}\begin{pmatrix}-1\\1\\2\\0\end{pmatrix}$$

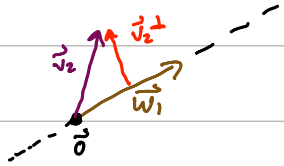
$$= \begin{pmatrix}1/3\\-1/3\\1/3\\1\end{pmatrix} \xrightarrow[\times 3]{} \begin{pmatrix}1\\-1\\1\\3\end{pmatrix}$$

$\perp$ basis: $\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\3\end{pmatrix} \right\}$

orthonormal basis: $\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\\0\end{pmatrix}, \frac{1}{\sqrt{12}}\begin{pmatrix}1\\-1\\1\\3\end{pmatrix} \right\}$

Why does Graham Schmitt process work:

- It produces k new vectors
- All vectors are $\perp$ to each other



$$\vec{w}_2 \cdot \vec{w}_1 = \underbrace{(\vec{v}_2 - \text{proj}_{w_1}(v_2))}_{\vec{v}_2^{\perp}} \cdot \vec{w}_1 = 0$$

- All the vectors are non-zero

$\rightarrow \{\vec{w}_1, \dots, \vec{w}_k\}$ are linearly independent

$\rightarrow$ basis because $k = \dim(W)$

<u>Def</u>: Given a subspace W of $\mathbb{R}^n$, the orthogonal complement of W is the set $W^{\perp} = \{\vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0 \ \forall \vec{w} \in W\}$

$\boxed{\text{e.g}}$

$W = \mathbb{R}^n \rightarrow W^{\perp} = \{\vec{0}\}$

$W = \{\vec{0}\} \rightarrow W^{\perp} = \mathbb{R}^n$

$\boxed{\text{e.g.}}$ $L = \text{span}(\vec{v})$, $\vec{v} \in \mathbb{R}^2$ non-$\vec{0}$

$\rightarrow L^{\perp} = \text{span}(\vec{w})$

for some non-$\vec{0}$ $\vec{w} \perp \vec{v}$

<u>Proposition</u>: $W \subset \mathbb{R}^n$ subspace $\rightarrow$ $W^{\perp}$ is also a subspace

<u>Proof</u>:

- 1^{st} observe $\forall \vec{w} \in W$, $\vec{0} \cdot \vec{w} = 0 \rightarrow \vec{0} \in W^{\perp}$
- 2^{nd} if $\vec{v} \in W^{\perp}$ and $c \in \mathbb{R}$, then $\forall \vec{w} \in W$ we have $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w}) = c \cdot 0 = 0$
  so $c\vec{v} \in W^{\perp}$
- 3^{rd}: if $\vec{v}_1, \vec{v}_2 \in W^{\perp}$ similary argue $\vec{v}_1 + \vec{v}_2 \in W^{\perp}$

$\therefore W^{\perp}$ is a subspace

Proposition: If $W = \text{span}(\vec{w}_1, \dots, \vec{w}_2) \subset \mathbb{R}^n$ then $\vec{v} \in W^\perp$ i.f.f. $\vec{v}\cdot\vec{w} = \dots = \vec{v}\cdot\vec{w}_K = 0$

Proof: **Idea** $\vec{w} \in W \to \vec{w} = c\vec{w}_1 + \dots + c_K\vec{w}_K \longrightarrow \vec{v}\cdot\vec{w} = c_1\,\vec{v}\cdot\vec{w}_1 + \dots + c_K\,\vec{v}\cdot\vec{w}_K$

$\boxed{\text{Ex}}$ $W = \{ \vec{x} \in \mathbb{R}^3 : 2x_1 - x_2 - x_3 = 0 \}$

$$= \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

Is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \in W^\perp$?

Check $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = 3$

$\boxed{\text{No!}}$

Is $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \in W^\perp$?

$\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = 0$ ✓

$\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 0$ ✓

$\boxed{\text{yes!}}$

Orthogonal Decomposition Theorem: If $W \subset \mathbb{R}^n$ is a unique subspace and $\vec{v} \in \mathbb{R}^n$ a vector, then $\exists$ unique vectors $\vec{v}_{\parallel} \in W$ and $\vec{v}_{\perp} \in W^{\perp}$ such that $\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$. In fact if $\beta = \{\vec{w}_1, \dots, \vec{w}_k\}$ is an $\perp$ basis for W then $\boxed{\vec{v}_{\parallel} = \text{proj}_{\vec{w}_1}(\vec{v}_1) + \dots + \text{proj}_{\vec{w}_k}(\vec{v})}$

- We call $\vec{v}_{\parallel}$ the orthogonal projection of $\vec{v}$ onto W and write $\text{proj}_W(\vec{v}) = \vec{v}_{\parallel}$

$\boxed{\text{Ex}}$ What is $\text{proj}_W(\vec{e}_1)$ where $W = \{\vec{x} \in \mathbb{R}^3 : 2x_1 + x_2 - x_3 = 0\}$?

Last time we found an $\perp$ basis $\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right\}$

Solution is in online notes